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Decision Support

Some induced ordered weighted averaging operators and their use for solving group decision-making problems based on fuzzy preference relations

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Abstract

In [R.R. Yager, D.P. Filev, Operations for granular computing: Mixing words and numbers, in: Proceedings of the FUZZ-IEEE World Congress on Computational Intelligence, Anchorage, 1998, pp. 123–128] Yager and Filev introduced the Induced Ordered Weighted Averaging (IOWA) operator. In this paper, we provide some IOWA operators to aggregate fuzzy preference relations in group decision-making (GDM) problems. These IOWA operators when guided by fuzzy linguistic quantifiers allow the introduction of some semantics or meaning in the aggregation, and therefore allow for a better control over the aggregation stage developed in the resolution process of the GDM problems. In particular, we present the Importance IOWA (I-IOWA) operator, which applies the ordering of the argument values based upon the importance of the information sources; the Consistency IOWA (C-IOWA) operator, which applies the ordering of the argument values based upon the consistency of the information sources; and the Preference IOWA (P-IOWA) operator, which applies the ordering of the argument values based upon the relative preference values associated to each one of them. We provide a procedure to deal with 'ties' in respect to the ordering induced by the application of one of these IOWA operators; it consists of a sequential application of the above IOWA operators. We also present a selection process for GDM problems based on the concept of fuzzy majority and the above three IOWA operators. Finally, we analyse the reciprocity and consistency properties of the collective fuzzy preference relations obtained using IOWA operators.

Keywords: Aggregation; Fuzzy linguistic quantifier; Group decision-making; OWA operator; IOWA operator

1. Introduction

There exist many opportunities to apply fuzzy sets theory in decision-making. For example, it can be used either to translate imprecise and vague information in the problem specification into fuzzy relationships (fuzzy

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objectives, fuzzy constraints, fuzzy preferences, etc.) or to design fuzzy tools for decision processes to establish preference orderings of alternatives. Several authors have provided interesting results on group decision-making (GDM) or social choice theory and multi-criteria decision-making (MCDM) with the help of fuzzy sets theory [7,9,12,14,18,19,31,32]. In all these decision-making problems, procedures have been established to combine opinions about alternatives related to different points of view.

In this paper the context of GDM is considered. We suppose that we have a group of experts, $E = \{e_1, \ldots, e_m\}$, which provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \ldots, P^m\}$, $P^k = [p^k_{ij}]$, $p^k_{ij} \in [0, 1]$, which are additive reciprocal, i.e., $p^k_{ij} + p^k_{ji} = 1,\ \forall i, j, k$.

MCDM are normally solved applying two steps [7,19]: *aggregation* and *exploitation*. Clearly, both steps are also applicable to GDM as has been shown in [1,12]. The aggregation step of a GDM problem consists in combining the experts' individual preferences into a group collective one in such a way that it summarizes or reflects the properties contained in all the individual preferences. In the literature, we can find different aggregation operators to aggregate preferences [7,29]. The exploitation phase transforms the global information about the alternatives into a global ranking of them. This can be done in different ways, the most common one being the use of a ranking method to obtain a score function.

GDM problems are roughly classified into two groups: homogeneous and heterogeneous [9,11]. A GDM problem is classified as homogeneous when no explicit importance degrees are provided or associated to the experts, in which case we say that all the experts are equally important, and is classified as heterogeneous in another case. However, we make note that, even in the case that no explicit importance degrees are provided, many GDM problems which are classified as homogeneous should be treated as heterogeneous when examined in depth. Indeed, in many cases an explicit presence of the importance degrees associated to the experts is not necessary to be absolutely sure that all of them should not be treated as equally important. This is specially true, for instance, once the experts have provided information on the particular matter to solve, in which case this information can be used as a mean to discriminate them as not equally important. In these cases, it may be reasonable to give more importance to the experts that provide the more *consistent* information.

The general procedure for the inclusion of the importance degrees in the aggregation process involves the transformation of the preference values under the importance degree to generate new values. This activity is carried out by means of a transformation function. Examples of such a function used in these cases include the minimum operator [9], the exponential function [20], or a t-norm operator [33]. An alternative way of implementing these importance degrees in the resolution process of a GDM problem is by using them to induce the ordering of the preference values prior to their aggregation, i.e. to use an Induced Ordered Weighted Averaging (IOWA) operator.

In this paper, we propose this last use of the importance degrees and introduce three particular cases of IOWA operators to aggregate fuzzy preference relations:

- The first one is the Importance IOWA (I-IOWA) operator, which applies the ordering of the argument values based upon the importance of the information sources. Obviously, the I-IOWA operator may be applied just when the GDM problem is classified as heterogeneous.
- If the GDM problem is classified as homogeneous, in which case the I-IOWA can not be applied, once the experts provide their preference relations we analyse the consistency of them and proceed to classify the experts from most consistent to least consistent. Thus, in the case of homogeneous GDM problems we define the Consistency IOWA (C-IOWA) operator, which applies the ordering of the argument values based upon the consistency of the information sources.
- Finally, we define the Preference IOWA (P-IOWA) operator, which applies the ordering of the argument values based upon the relative preference values associated to each one of them.

We also provide a different procedure to the one proposed by Yager and Filev in [27] for dealing with 'ties' in respect to the ordering induced by the application of one of these IOWA operators. This procedure consists of a repeated application of the above IOWA operators. Finally, we show that, in general, IOWA operators maintain the reciprocity property of fuzzy preference relations as well as the consistency property after aggregation is carried out in the resolution process.

In order to do this, this paper is set out as follows. In Section 2, we summarise the basic operators used in this study: the OWA and IOWA operators. In Section 3, we define the three particular cases of IOWA operators to aggregate fuzzy preference relations in GDM problems: the I-IOWA, C-IOWA and P-IOWA operators. In Section 4, we present a selection process for GDM problems based on the concept of fuzzy majority and the above three IOWA operators. Also, a procedure is proposed to deal with 'ties' that could appear in the ordering induced by the application of one of the above IOWA operators in the aggregation of fuzzy preference relations. In Section 5, we show that, in general, the collective preference relation obtained by applying IOWA operators verifies the reciprocity property as well as the consistency property when the aggregated individual preference ones do. Finally, in Section 6 we draw our conclusions.

2. Preliminaries: OWA and IOWA operators

In this section we start by providing a summary of the concepts of OWA and IOWA operators, which will be used throughout the paper.

2.1. The OWA operator

In [1] Chiclana et al. considered GDM problems where the information about the alternatives was represented using *fuzzy preference relations* and a *fuzzy majority* guided choice scheme was designed. This choice scheme is based on the OWA operator [22] and follows two steps to achieve a final decision from the synthesis of preference intensity degrees of the majority of experts: (i) `aggregation` and (ii) `exploitation`.

Definition 1. An OWA operator of dimension n is a function $\phi : \mathbb{R}^n \to \mathbb{R}$, that has associated a set of weights or weighting vector $W = (w_1, \dots, w_n)$ to it, so that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, and is defined to aggregate a list of values $\{p_1, \dots, p_n\}$ according to the following expression:

$$\phi(p_1, \dots, p_n) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)},$$

being $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ a permutation such that $p_{\sigma(i)} \geqslant p_{\sigma(i+1)}$, $\forall i = 1, \dots, n-1$, i.e., $p_{\sigma(i)}$ is the ith highest value in the set $\{p_1, \dots, p_n\}$.

An issue in the definition of the OWA operator is how to obtain the associated weighting vector. In [22], Yager proposed two ways to obtain it. The first approach is to use some kind of learning mechanism using some sample data; and the second approach is to try to give some semantics or meaning to the weights. The latter allowed applications in the area of quantifier guided aggregations [21].

In the process of quantifier guided aggregation, given a collection of n criteria represented as fuzzy subsets of the alternatives X, the OWA operator has been used to implement the concept of fuzzy majority in the aggregation phase by means of a *fuzzy linguistic quantifier* [30] which indicates the proportion of satisfied criteria 'necessary for a good solution' [23]. This implementation is done by using the quantifier to calculate the OWA weights.

Given a function $Q : [0,1] \to [0,1]$ such that $Q(0) = 0$, $Q(1) = 1$ and if $x > y$ then $Q(x) \geqslant Q(y)$, an OWA aggregation guided by this function can be obtained as [22]:

$$\phi_Q(p_1, \dots, p_n) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)},$$

being $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ a permutation such that $p_{\sigma(i)} \geqslant p_{\sigma(i+1)}$, $\forall i = 1, \dots, n-1$, i.e., $p_{\sigma(i)}$ is the ith largest value in the set $\{p_1, \dots, p_n\}$; and

$$w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right), \quad i = 1, \dots, n. \tag{1}$$

These function are called Basic Unit-interval Monotone (BUM) functions in [26] and 'are particularly useful in situations in which the imperative guiding the OWA aggregation is expressed linguistically by a quantifier'. We make note that in [23] BUM functions are called Regular Increasing Monotone (RIM) quantifiers.

Example 1. Suppose three experts provide the following fuzzy preference relations on a set of three alternatives

$$P^1 = \begin{pmatrix} 0.5 & 0.75 & 0.87 \\ 0.25 & 0.5 & 0.66 \\ 0.13 & 0.34 & 0.5 \end{pmatrix}; \quad P^2 = \begin{pmatrix} 0.5 & 0.66 & 0.94 \\ 0.34 & 0.5 & 0.87 \\ 0.06 & 0.13 & 0.5 \end{pmatrix}; \quad P^3 = \begin{pmatrix} 0.5 & 0.66 & 0.75 \\ 0.34 & 0.5 & 0.66 \\ 0.25 & 0.34 & 0.5 \end{pmatrix}.$$

Following the choice scheme defined in [1], if we aggregate them by using an OWA operator guided by the fuzzy linguistic quantifier "*most of*" defined by $Q(r) = r^{1/2}$ [23], whose corresponding weighting vector using (1) is $(0.58, 0.24, 0.18)$, then we have the following collective preference relation:

$$P^c = \phi_{\text{most}}(P^1, P^2, P^3) = \begin{pmatrix} 0.5 & 0.71 & 0.89 \\ 0.32 & 0.5 & 0.78 \\ 0.19 & 0.3 & 0.5 \end{pmatrix},$$

whose elements can be interpreted as the preference degree of one alternative over another for most of the experts.

We make note that this type of aggregation 'is very strongly dependent upon the weighting vector used' [23], and consequently also upon the function expression used to represent the fuzzy linguistic quantifier. The particular RIM function used in this example guarantees that all the experts contribute to the final aggregated value because it is a strictly increasing function. Moreover, the higher the ranking of a value, the higher the weighting value associated to it. Therefore, this RIM function seems to be adequate for those decision-making problems where the importance or consistency degrees of the experts are used to induce the order of the values to be aggregated.

2.2. The IOWA operator

In [15] Mitchell and Estrakh described a modified OWA operator in which the input arguments are not rearranged according to their values but rather using a function of the arguments. Inspired by this work, Yager and Filev introduced in [27] a more general type of OWA operator, which they named the Induced Ordered Weighted Averaging (IOWA) operator:

Definition 2. An IOWA operator of dimension n is a function $\Phi_W : (\mathbb{R} \times \mathbb{R})^n \to \mathbb{R}$, to which a set of weights or weighting vector is associated, $W = (w_1, \ldots, w_n)$, such that $w_i \in [0, 1]$ and $\Sigma_i w_i = 1$, and it is defined to aggregate the set of second arguments of a list of n 2-tuples $\{\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle\}$ according to the following expression:

$$\Phi_W(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle) = \sum_{i=1}^{n} w_i \cdot p_{\sigma(i)},$$

being $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ a permutation such that $u_{\sigma(i)} \geqslant u_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, i.e., $\langle u_{\sigma(i)}, p_{\sigma(i)}\rangle$ is the 2-tuple with $u_{\sigma(i)}$ the ith highest value in the set $\{u_1, \ldots, u_n\}$.

In the above definition the reordering of the set of values to aggregate, $\{p_1, \ldots, p_n\}$, is induced by the reordering of the set of values $\{u_1, \ldots, u_n\}$ associated to them, which is based upon their magnitude. Due to this use of the set of values $\{u_1, \ldots, u_n\}$, Yager and Filev called them the values of an order inducing variable and $\{p_1, \ldots, p_n\}$ the values of the argument variable [27,28,24,26]. As we have mentioned, the main difference between the OWA operator and the IOWA operator resides in the reordering step of the argument variable. In the case of OWA operator this reordering is based upon the magnitude of the values to be aggregated, while in the case of IOWA operator an order inducing variable is used as the criterion to induce that reordering. Obviously, an immediate consequence of Definition 2 is that if the order inducing variable is the argument variable then the IOWA operator is reduced to the OWA operator.

Another important distinction between OWA and IOWA aggregation processes arises when there is a tie in the ordering operation. Ties in the arguments to be aggregated do not present a problem in OWA aggregation processes; the same result is obtained no matter the order in which the tied arguments are placed. This is not the case for IOWA aggregation processes, as the following example illustrates. Consider the aggregation of the objects $\{\langle 5,1\rangle, \langle 3,0.5\rangle, \langle 8,0.6\rangle, \langle 5,0.4\rangle\}$ using the weighting vector $W = (0.4, 0.3, 0.2, 0.1)$. There is a tie between $\langle 5,1\rangle$ and $\langle 5,0.4\rangle$ with respect to the order inducing variable. If the tie is broken by selecting $\langle 5,1\rangle$ ahead of $\langle 5,0.4\rangle$ then the result would be different to the one obtained if $\langle 5,0.4\rangle$ is put ahead of $\langle 5,1\rangle$. Yager and Filev's approach to the case of ties in the order inducing process is to replace the arguments of all the tied pairs by their average [27]. Using this policy, in the above example both arguments 1 and 0.4 would be replaced by 0.7.

In the following result we summarise some of the properties that IOWA operators verify:

Proposition 1. *The IOWA operator satisfies the following properties*:

1. *It is commutative*:

$$\Phi_W(\langle u_{\sigma_1(1)}, p_{\sigma_1(1)}\rangle, \ldots, \langle u_{\sigma_1(n)}, p_{\sigma_1(n)}\rangle) = \Phi_W(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle),$$

being σ_1 *a permutation of the set* $\{1, \ldots, n\}$ *and* $(\langle u_{\sigma_1(1)}, p_{\sigma_1(1)}\rangle, \ldots, \langle u_{\sigma_1(n)}, p_{\sigma_1(n)}\rangle)$ *a reordering of the set of* 2-*tuples* $(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle)$.

2. *It is an or–and operator*, *i.e.*, *it is located between the minimum and the maximum of the arguments to be aggregated*:

$$\min_i\{p_i\} \leqslant \Phi_W(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle) \leqslant \max_i\{p_i\}.$$

3. *It is idempotent with respect to the argument variable*:

$$\Phi_W(\langle u_1, p\rangle, \ldots, \langle u_n, p\rangle) = p.$$

4. *It is increasingly monotonous with respect to the argument variable when the order inducing values are unchanged*:

$$\Phi_W(\langle u_1, p_1\rangle, \ldots, \langle u_n, p_n\rangle) \leqslant \Phi_W(\langle u_1, q_1\rangle, \ldots, \langle u_n, q_n\rangle) \quad \text{if } p_i \leqslant q_i \ \forall i.$$

5. *The IOWA operator is reduced to the Average or Arithmetic Mean* (*AM*) *operator when* $w_i = \frac{1}{n}, \ \forall i$.
6. *The IOWA operator is reduced to the Weighted Averaging* (*WA*) *operator when the* 2-*tuples have the following expression* $\langle f(n-i+1), a_i\rangle$, *being f a strictly increasing function.*
7. *The IOWA operator is reduced to the OWA operator when the* 2-*tuples have the following expression* $\langle f(p_i), p_i\rangle$, *being f a strictly increasing function.*
Additionally, *we have the two following cases*:
   - *If* $W^* = (1, 0, \ldots, 0)$ *the IOWA operator is reduced to the maximum operator*:

   $$\Phi_{W^*}(\langle f(p_1), p_1\rangle, \ldots, \langle f(p_n), p_n\rangle) = \max_i\{p_i\}.$$

   - *If* $W_* = (0, \ldots, 0, 1)$ *the IOWA operator is reduced to the minimum operator*:

   $$\Phi_{W_*}(\langle f(p_1), p_1\rangle, \ldots, \langle f(p_n), p_n\rangle) = \min_i\{p_i\}.$$

For a detailed proof of the above list of properties and uses of the IOWA operators the reader should consult [16,24–28].

Note 1. In this paper we will focus on the aggregation of numerical preferences, which is why we assume that the nature of the first argument of the IOWA operators is also numeric, although it could be linguistic [25–28].

Note 2. In the case of using an IOWA operator in the aggregation phase of a GDM problem, the concept of fuzzy majority can also be implemented by means of the *fuzzy linguistic quantifiers* [30]. When a fuzzy linguistic quantifier Q is used to compute the weights of the IOWA operator Φ, then it is symbolized by Φ_Q.

Example 2. If we want to aggregate the set of 2-tuples $\{\langle 0.65, 0.87\rangle, \langle 0.13, 0.94\rangle, \langle 0.22, 0.75\rangle\}$, using again the fuzzy linguistic quantifier "*most of*" of Example 1, then we obtain

$$\Phi_{\text{most}}(\langle 0.65, 0.87\rangle, \langle 0.13, 0.94\rangle, \langle 0.22, 0.75\rangle) = 0.58 \cdot 0.87 + 0.24 \cdot 0.75 + 0.18 \cdot 0.94 = 0.85.$$

3. Some IOWA operators to aggregate fuzzy preference relations

In [1] we designed a *fuzzy majority* guided choice scheme to achieve a final decision from the synthesis of preference intensity degrees of the majority of experts. This choice scheme is based on the quantifier guided aggregation operator, the OWA operator [22], which implements the concept of fuzzy majority in the aggregation phase by means of the *fuzzy quantifiers* [30] used to calculate its weighting vector.

A fundamental aspect of the OWA operators is the reordering of the arguments to be aggregated, based upon the magnitude of their respective values, which allows us to give importance to values in opposition to the Weighted Average (WA) operators which compute an aggregate value taking into account the reliability of the sources of information [29]. However, it is clear that a set of values can be reordered in a different way to the one used by the OWA operators. To do this, a criterion has to be defined to induce a specific ordering of the arguments to be aggregated before a WA operator can be applied. This is the idea upon which Yager and Filev based the definition of the IOWA operator [27]. While the OWA operators order the arguments by their value, the IOWA operators induce their ordering by using an additional variable or criterion, called the order inducing variable. In fact, the OWA operator as well as the WA operator are included in the more general class of IOWA operators [28]. This means that the IOWA operators allow us to take control of the aggregation stage of any GDM problem in the sense that importance can be given to the magnitude of the values to be aggregated as the OWA operators do or to the information sources as the WA operators do.

In this section we present three special cases of IOWA operators for GDM problems with fuzzy preference relations. These IOWA operators when guided by fuzzy quantifiers allow the introduction of some semantics or meaning in the aggregation, and therefore allow for better control over the aggregation stage:

- The first two (I-IOWA and C-IOWA) act as the WA operator because the aggregation is based upon the reliability of the information sources, while
- the third one (P-OWA) acts as the OWA operator because the ordering of the argument values is based upon a relative magnitude associated to each one of them.

The first one is proposed to be used in heterogeneous GDM problems, while the other two can be applied both in homogeneous and heterogeneous GDM problems.

3.1. The importance induced ordered weighted averaging (I-IOWA) operator

In many cases, each expert $e_k \in E$ is assigned an *importance degree* to him/her. We can assume without loss of generality that $u_i \in [0,1]\ \forall i$, and that there is some i such that $u_i = 1$. We can always obtain these two conditions by dividing all importance degrees by their maximum. Thus, *importance degree* can be interpreted as a fuzzy set membership function, $\mu_I : E \to [0,1]$, in such a way that $\mu_I(e_k) = u_k \in [0,1]$ denotes the importance degree of the opinion provided by the expert e_k. When this is the case, we call this a heterogeneous GDM problem [3,5,4,8,9,11,17].

The general procedure for the inclusion of importance weight values in the aggregation process involves the transformation of the preference values, p_{ij}^k, under the importance degree u_k to generate a new value, $\bar{p}_{ij}^k$, and then aggregate these new values using an aggregation operator. Obviously, the form of transformation function depends on the characteristic of the aggregation operator. When using the average operator, the approach for including importance degrees is to use the weighted average, i.e., $g(p_{ij}^k, u_k) = \frac{n}{S(n)} \cdot u_k \cdot p_{ij}^k$, with $S(n) = \sum_{l=1}^{n} u_l$ [3,17]. When using the min-type aggregation operator, the function $g(p_{ij}^k, u_k) = \max(p_{ij}^k, u_k)$ was proposed to limit the right of veto [5]. Yager in [27] notes that "when calculating the min, it is the lower scores that play the more significant role", and he proposes to use the transformation function $g(p_{ij}^k, u_k) = \max(p_{ij}^k, 1 - u_k)$, as in [6]. In general, in this case the use of a t-conorm s is suggested, i.e., $g(p_{ij}^k, u_k) = s(p_{ij}^k, 1 - u_k)$. A different method for including importance degrees in the min-type aggregation not included in the preceding class is the use of the exponential function $g(p_{ij}^k, u_k) = (p_{ij}^k)^{u_k}$, suggested by Yager in [20]. On the other hand, when using a max-type aggregation, importance degrees are included by using a t-norm, for example $g(p_{ij}^k, u_k) = \min(p_{ij}^k, u_k)$ [5]. An alternative approach not following the t-norm function to include importance degrees when using the max-type aggregation is the use of the transformation function $g(x,y) = 1 - (1 - x)^y$, which was also proposed by Yager [20].

In the area of quantifier guided aggregations, Yager in [23] presents a procedure to evaluate the overall satisfaction of Q important criteria (experts) by the alternative x_i. In this procedure, once the satisfaction values to be aggregated have been ordered, the weighting vector associated to an OWA operator using a linguistic quantifier Q is calculated following the expression:

$$w_k = Q\left(\frac{S(k)}{S(n)}\right) - Q\left(\frac{S(k-1)}{S(n)}\right), \tag{2}$$

being $S(k) = \sum_{l=1}^{k} u_{\sigma(l)}$, and σ the permutation used to produce the ordering of the values to be aggregated. This approach for the inclusion of importance degrees associates a zero weight to those experts with zero importance degree.

This procedure was extended by Yager to the case of induced aggregation [26]. In this case, each component in the aggregation consists of a triple (p_{ij}^k, u_k, v_k) : p_{ij}^k is the argument value to aggregate, u_k is the importance weight value associated to p_{ij}^k, and v_k is the order inducing value. In this case the aggregation is

$$\Phi_Q(p_{ij}^1, \ldots, p_{ij}^n) = \sum_{k=1}^{n} w_k \cdot p_{ij}^{\sigma(k)},$$

with

$$w_k = Q\left(\frac{S(k)}{S(n)}\right) - Q\left(\frac{S(k-1)}{S(n)}\right), \tag{3}$$

where $S(k) = \sum_{l=1}^{k} u_{\sigma(l)}$, and σ is the permutation such that $v_{\sigma(k)}$ in $(p_{ij}^{\sigma(k)}, u_{\sigma(k)}, v_{\sigma(k)})$ is the kth largest value in the set $\{v_1, \ldots, v_n\}$.

In our case, we propose to use the importance degrees associated to each one of the experts both as a weight associated to the argument to aggregate and as the order inducing values ($v_i = u_i$). Thus, the ordering of the preference values is first induced by the ordering of the experts from most to least important one, and the weights of the IOWA operator is obtained by applying the above Eq. (3), which reduces to:

$$w_k = Q\left(\frac{S(k)}{S(n)}\right) - Q\left(\frac{S(k-1)}{S(n)}\right), \tag{4}$$

where $S(k) = \sum_{l=1}^{k} u_{\sigma(l)}$, and σ is the permutation such that $u_{\sigma(k)}$ in $(p_{ij}^{\sigma(k)}, u_{\sigma(k)})$ is the kth largest value in the set $\{u_1, \ldots, u_n\}$. We call this importance degree based IOWA operator as the Importance IOWA (I-IOWA) operator and denote it as Φ_W^I.

Definition 3. If a set of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \ldots, P^m\}$, and each expert e_k has an importance degree, $\mu_I(e_k) \in [0, 1]$, associated to him or her, then an I-IOWA operator of dimension n, Φ_W^I, is an IOWA operator whose set of order inducing values is the set of importance degrees.

Example 3. Suppose that the importance degrees of the set of three experts of Example 1 are the following $I = (2.12, 1.01, 1.37)$. Using the fuzzy linguistic quantifier "*most of*" and expression (4), the collective fuzzy preference relation is

$$P^c = \Phi_{\text{most}}^I\left(\langle 2.12, P^1\rangle, \langle 1.01, P^2\rangle, \langle 1.37, P^3\rangle\right) = 0.69 \cdot P^1 + 0.19 \cdot P^3 + 0.12 \cdot P^2 = \begin{pmatrix} 0.5 & 0.72 & 0.86 \\ 0.28 & 0.5 & 0.69 \\ 0.14 & 0.31 & 0.5 \end{pmatrix},$$

whose elements can be considered as the preference of one alternative over another for most of the more important experts.

Note 3. It is worth noting that our approach to the importance weight treatment using formula (4) via the I-IOWA operator is computationally more efficient than the original OWA based approach. In our approach the weights are computed once for the whole aggregation process while in the latter case the weights have to be computed for each cell of the resulting fuzzy preference relation.

Note 4. We consider that the importance degrees should be implemented in an aggregation process in such a way that the effect from those experts who are less important is reduced or mitigated. We also propose the importance degrees to be used to induce the order of the experts prior to the aggregation, and therefore the above is obtained if the linguistic quantifier Q verifies that the higher the importance of an expert the higher the weighting value of that expert in the aggregation, i.e.:

$$u_{\sigma(1)} \geqslant u_{\sigma(2)} \geqslant \cdots \geqslant u_{\sigma(n)} \geqslant 0 \Rightarrow w_1 \geqslant w_2 \geqslant \cdots \geqslant w_n \geqslant 0. \tag{5}$$

Yager in [23] considers the parameterized family of RIM quantifiers

$$Q(r) = r^a, \quad a \geqslant 0$$

and the particular function with $a = 2$ to represent the fuzzy linguistic quantifier "most of". This strictly increasing and convex function does not verify the above as the following example shows. If we had the following importance degrees (0.9, 0.75, 0.3, 0.25, 0.2, 0.15, 0.1) then formula (2) with $Q(r) = r^2$ leads to the weighting vector (0.12, 0.26, 0.16, 0.14, 0.13, 0.11, 0.08).

For simplicity we will drop the σ symbol in what follows. Assuming $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_n \geqslant 0$, we have the following equivalence:

$$w_i \geqslant w_{i+1} \iff Q(T_i) - Q(T_{i-1}) \geqslant Q(T_{i+1}) - Q(T_i) \iff Q(T_i) \geqslant \frac{Q(T_{i-1}) + Q(T_{i+1})}{2},$$

where $T_i = \frac{S(i)}{S(n)}$, and $S(i) = \sum_{l=1}^{i} u_l,\ S(0) = 0$. Also, it is true that

$$S_{i-1} + S_{i+1} = \sum_{l=1}^{i-1} u_l + \sum_{l=1}^{i+1} u_l = 2 \cdot \sum_{l=1}^{i-1} u_l + u_i + u_{i+1} \leqslant 2 \cdot \sum_{l=1}^{i-1} u_l + u_i + u_i = 2 \cdot S_i$$

and therefore

$$T_i \geqslant \frac{T_{i-1} + T_{i+1}}{2}.$$

Clearly, in our case any increasing concave linguistic quantifier Q will verify (5). Indeed, for being Q increasing we have first

$$Q(T_i) \geqslant Q\left(\frac{T_{i-1} + T_{i+1}}{2}\right).$$

Concavity of Q implies that

$$Q\left(\frac{T_{i-1} + T_{i+1}}{2}\right) \geqslant \frac{Q(T_{i-1}) + Q(T_{i+1})}{2}.$$

For the parameterized family of RIM quantifiers $Q(r) = r^a$, $a \geqslant 0$, if $a \in [0,1]$ then function $Q(x) = x^a$ is concave and fit for our purpose. Also, for $a \in [0, 1]$ the computed weighting vector presents an orness measure $\text{orness}(W) = \frac{1}{1+a} \geqslant 0.5$, and consequently, the lower a the closer the IOWA aggregation guided by Q will be to the maximum aggregation operator.

Note 5. The following example shows that formula (1) for calculating the I-IOWA weights does not capture the majority concept "*most of the important*", while formula (4) does. Suppose that we want to aggregate the first arguments of the following list of 2-tuples

$$\{\langle 0.2, 0.3\rangle, \langle 0.75, 0.7\rangle, \langle 0.3, 0.1\rangle, \langle 0.2, 0.15\rangle, \langle 0.3, 0.25\rangle, \langle 0.7, 0.9\rangle, \langle 0.2, 0.2\rangle\},$$

where the second arguments are importance degrees that induce the following ordering:

$$\sigma(1) = 6 \quad \sigma(2) = 2 \quad \sigma(3) = 1 \quad \sigma(4) = 5 \quad \sigma(5) = 7 \quad \sigma(6) = 4 \quad \sigma(7) = 3.$$

Using formula (4) with $Q(r) = r^{1/2}$ we get the following weights:

$$w_1 = 0.378 \quad w_2 = 0.157 \quad w_3 = 0.12 \quad w_4 = 0.101 \quad w_5 = 0.089 \quad w_6 = 0.081 \quad w_7 = 0.074,$$

while using formula (1) with $Q(r) = r^{1/2}$ we get the following weights:

$$w_1 = 0.588 \quad w_2 = 0.196 \quad w_3 = 0.07 \quad w_4 = 0.055 \quad w_5 = 0.041 \quad w_6 = 0.03 \quad w_7 = 0.019.$$

In both cases the higher the importance degree the higher the weight, as a consequence of the linguistic quantifier being concave. However, it is apparent that this effect is magnified in the second case as a result of taking into account the importance degrees not just for inducing the order but for computing the aggregation weights as well. Indeed, using formula (1) the same weights would have been obtained if for example the importance values had been

$$u_1 = 0.3 \quad u_2 = 0.7 \quad u_3 = 0.01 \quad u_4 = 0.015 \quad u_5 = 0.025 \quad u_6 = 0.9 \quad u_7 = 0.02.$$

This is not the case when using formula (4), where the new weights

$$w_1 = 0.686 \quad w_2 = 0.225 \quad w_3 = 0.081 \quad w_4 = 0.006 \quad w_5 = 0.005 \quad w_6 = 0.004 \quad w_7 = 0.003$$

reflect the change in the importance degrees. This effect is also reflected in the aggregated value: using formula (1) we get 0.493 in both cases, while the values we get using formula (4) are 0.609 and 0.663 respectively, which reflect better the values of the most important experts.

3.2. The consistency induced ordered weighted averaging (C-IOWA) operator

When the experts have equal importance, i.e., in a homogeneous GDM problem, the I-IOWA operator is reduced to the Average Mean (AM) operator. However, in a homogeneous situation, each expert can always have a consistency index value associated to him or her. Usually, for each expert this consistency index value is obtained by analysing his or her fuzzy preference relation, and then, we can use it as the order inducing variable in the aggregation of preferences by means of IOWA operators.

In decision-making problems based on fuzzy preference relations, the study of consistency is associated with the study of the *transitivity property*. In [10], Herrera-Viedma et al. gave a characterization of the consistency property defined by the additive transitivity property of a fuzzy preference relation $P^k = (p^k_{ij})$:

$$p^k_{ij} + p^k_{jl} + p^k_{li} = \frac{3}{2}, \quad \forall i, j, l \in \{1, \ldots, n\}.$$

Using this characterization method, a procedure was given to construct a consistent fuzzy preference relation $\widetilde{P}^k$ from a non-consistent fuzzy preference relation P^k.

Summarising, the method to construct a consistent reciprocal fuzzy preference relation $\widetilde{P}$ on $X = \{x_1, \ldots, x_n, n \geqslant 2\}$ from $n-1$ preference values $\{p_{12}, p_{23}, \ldots, p_{n-1n}\}$ presents the following steps:

1. $\widehat{P} = (\widehat{p}_{ij})$ such that:

$$\widehat{p}_{ij} = \begin{cases} p_{ij} & \text{if } i \leqslant j \leqslant i+1, \\ \left(p_{ii+1} + p_{i+1i+2} \cdots + p_{j-1j}\right) - \frac{j-(i+1)}{2} & \text{if } j > i+1, \\ 1 - \widehat{p}_{ji}, & \text{if } j < i. \end{cases}$$

We make note that the matrix $\widehat{P}$ could have entries not in the interval $[0, 1]$, but in an interval $[-a, 1+a]$, being $a = |\min\{\widehat{p}_{ij}; \widehat{p}_{ij} \in \widehat{P}\}|$. In such a case, we would need to transform the values obtained via a transformation function which preserves reciprocity and additive consistency, that is a function $f : [-a, 1+a] \to [0, 1]$, verifying
(a) $f(-a) = 0$
(b) $f(1+a) = 1$

(c) $f(x) + f(1 - x) = 1, \forall x \in [-a, 1 + a]$
(d) $f(x) + f(y) + f(z) = \frac{3}{2}, \forall x, y, z \in [-a, 1 + a]$ such that $x + y + z = \frac{3}{2}$.
A simple function verifying 1 and 2 takes the form $f(x) = \varphi \cdot x + \beta$, being $\varphi, \beta \in \mathbb{R}$. This function is

$$f(x) = \frac{1}{1 + 2a} \cdot x + \frac{a}{1 + 2a} = \frac{x + a}{1 + 2a},$$

which verifies (c)

$$f(x) + f(1 - x) = \frac{x + a}{1 + 2a} + \frac{1 - x + a}{1 + 2a} = \frac{x + a + 1 - x + a}{1 + 2a} = 1$$

and when $x + y + z = \frac{3}{2}$

$$f(x) + f(y) + f(z) = \frac{x + a}{1 + 2a} + \frac{y + a}{1 + 2a} + \frac{z + a}{1 + 2a} = \frac{x + y + z + 3a}{1 + 2a} = \frac{3/2 + 3a}{1 + 2a} = \frac{3}{2}$$

verifies (d).
2. The consistent fuzzy preference relation $\widetilde{P}$ is obtained as $\widetilde{P} = f(\widehat{P})$.

We make note that this method depends on the labelling of the alternatives and therefore, the resulting consistent fuzzy preference relation is not unique.

Let $\{\widetilde{P}_1, \ldots, \widetilde{P}_{n!}\}$ be the set of all possible consistent matrixes that can be obtained for a given matrix P according to the above method and $\widetilde{P}$ their average.

The distance between P^k and $\widetilde{P}^k$ can be used as a measure of the consistency of matrix P^k and hence of the expert who provided it:

$$CI^k = d(P^k, \widetilde{P}^k) = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} (p_{ij}^k - \widetilde{p}_{ij}^k)^2}.$$

The closer $1 - CI^k$ is to 1 the more consistent the information provided by the expert e^k, and thus more importance should be placed on that information. In other words, we could use these values to define the ordering of the argument values to be aggregated, in which case we would be implementing the concept of consistency in the aggregation process of our decision-making. This kind of aggregation process defines an IOWA operator that we call the Consistency IOWA (C-IOWA) operator and denote it as Φ_W^C.

Definition 4. If a set of experts, $E = \{e_1, \ldots, e_m\}$, provides preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \ldots, P^m\}$, then a C-IOWA operator of dimension n, Φ_W^C, is an IOWA operator whose set of order inducing values is the set of consistency index values, $\{1 - CI^1, \ldots, 1 - CI^m\}$, associated to the set of experts.

Example 4 [13]. Suppose a set of four alternatives $X = \{x_1, x_2, x_3, x_4\}$ and a set of four experts $E = \{e_1, e_2, e_3, e_4\}$, whose fuzzy preference relations on X are:

$$P^1 = \begin{pmatrix} 0.5 & 0.3 & 0.7 & 0.1 \\ 0.7 & 0.5 & 0.6 & 0.6 \\ 0.3 & 0.4 & 0.5 & 0.2 \\ 0.9 & 0.4 & 0.8 & 0.5 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.2 \\ 0.6 & 0.5 & 0.7 & 0.4 \\ 0.4 & 0.3 & 0.5 & 0.1 \\ 0.8 & 0.6 & 0.9 & 0.5 \end{pmatrix},$$

$$P^3 = \begin{pmatrix} 0.5 & 0.5 & 0.7 & 0 \\ 0.5 & 0.5 & 0.8 & 0.4 \\ 0.3 & 0.2 & 0.5 & 0.2 \\ 1 & 0.6 & 0.8 & 0.5 \end{pmatrix}, \quad P^4 = \begin{pmatrix} 0.5 & 0.4 & 0.7 & 0.8 \\ 0.6 & 0.5 & 0.4 & 0.3 \\ 0.3 & 0.6 & 0.5 & 0.1 \\ 0.2 & 0.7 & 0.9 & 0.5 \end{pmatrix}.$$

To compute the consistency indexes, for simplicity, we are going to obtain only one consistent matrix in each case, assuming the initial ordering of the alternatives. The corresponding consistent fuzzy preference relations obtained by applying the above procedure are:

$$\widetilde{P}^1 = \begin{pmatrix} 0.5 & 0.3 & 0.4 & 0.1 \\ 0.7 & 0.5 & 0.6 & 0.3 \\ 0.6 & 0.4 & 0.5 & 0.2 \\ 0.9 & 0.7 & 0.8 & 0.5 \end{pmatrix}, \quad \widetilde{P}^2 = \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.2 \\ 0.6 & 0.5 & 0.7 & 0.3 \\ 0.4 & 0.3 & 0.5 & 0.1 \\ 0.8 & 0.7 & 0.9 & 0.5 \end{pmatrix},$$
$$\widetilde{P}^3 = \begin{pmatrix} 0.5 & 0.5 & 0.8 & 0.5 \\ 0.5 & 0.5 & 0.8 & 0.5 \\ 0.2 & 0.2 & 0.5 & 0.2 \\ 0.5 & 0.5 & 0.8 & 0.5 \end{pmatrix}, \quad \widetilde{P}^4 = \begin{pmatrix} 0.5 & 5/12 & 1/3 & 0 \\ 7/12 & 0.5 & 5/12 & 1/12 \\ 2/3 & 7/12 & 0.5 & 1/6 \\ 1 & 11/12 & 5/6 & 0.5 \end{pmatrix}.$$

To obtain the elements of $\widetilde{P}^4$ is necessary to apply function $f(x) = (x + 0.1)/1.2$ because $a = |\widehat{p}^4_{14}| = |-0.1|$.

The consistency indexes associated to the experts are $\mathbf{1} - CI = (0.4, 0.86, 0.27, 0.23)$. The collective fuzzy preference relation obtained by using a C-IOWA operator guided by the same fuzzy linguistic quantifier "*most of*", with weighting vector (0.7, 0.15, 0.08, 0.07) calculated using the expression (4), is

$$P^c = \Phi^C_{\text{most}}(\langle 0.4, P^1\rangle, \langle 0.86, P^2\rangle, \langle 0.27, P^3\rangle, \langle 0.23, P^4\rangle) = \begin{pmatrix} 0.5 & 0.39 & 0.63 & 0.21 \\ 0.61 & 0.5 & 0.67 & 0.42 \\ 0.37 & 0.33 & 0.5 & 0.12 \\ 0.79 & 0.58 & 0.88 & 0.5 \end{pmatrix},$$

whose elements can be considered as the preference of one alternative over another for most of the more consistent experts.

Note 6. The additive transitivity property used does not include the ordinary transitivity for crisp preference relations. A modification of the definition of the additive transitivity will be studied in a future paper. Furthermore, we should point out that we use the additivity transitivity as just an example of the consistency notion to compute consistency index values. Therefore, any other consistency property that allows to obtain consistency indexes would be equally valid.

3.3. The preference induced ordered weighted averaging (P-IOWA) operator

If $P^k = (p^k_{ij})$ is a fuzzy preference relation on the set of alternatives $\{x_1, \ldots, x_n\}$ then the total sum of the elements of each row i, $\bar{p}^k_i = \sum_j p^k_{ij}$, can be interpreted as the total preference of the alternative x_i. The resulting value obtained by dividing an element of that row, p^k_{ir}, by $\bar{p}^k_i$, $\bar{p}^k_{ir} = \frac{p^k_{ir}}{\sum_j p^k_{ij}}$, can be interpreted as the relative preference contribution of that particular element to the total preference of the alternative x_i.

These relative preference values can be used as the order inducing values of an IOWA operator to aggregate a set of fuzzy preference relations. We call this a Preference IOWA (P-IOWA) operator and denote it as Φ^P_W.

Definition 5. If a set of experts, $E = \{e_1, \ldots, e_m\}$, provides preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \ldots, P^m\}$ then a P-IOWA operator of dimension n, Φ^P_W, is an IOWA operator whose set of order inducing values is the set of relative preference values associated to each one of the arguments to aggregate.

If a set of experts, $E = \{e_1, \ldots, e_m\}$, provides preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \ldots, P^m\}$ then a P-IOWA operator of dimension n, Φ^P_W, is an IOWA operator whose set of order inducing values is the set of relative preferences matrices, $\{\overline{P}^k = (\bar{p}^k_{ij});\ k = 1, \ldots, m\}$.

Example 5. Using the same data as in Example 4, the corresponding relative preference matrices, $\overline{P}^k$, are:

$$\overline{P}^1 = \begin{pmatrix} 0.31 & 0.19 & 0.44 & 0.06 \\ 0.29 & 0.21 & 0.25 & 0.25 \\ 0.21 & 0.29 & 0.36 & 0.14 \\ 0.35 & 0.15 & 0.31 & 0.19 \end{pmatrix}, \quad \overline{P}^2 = \begin{pmatrix} 0.29 & 0.24 & 0.35 & 0.12 \\ 0.27 & 0.23 & 0.32 & 0.18 \\ 0.31 & 0.23 & 0.38 & 0.08 \\ 0.29 & 0.21 & 0.32 & 0.18 \end{pmatrix},$$

$$\overline{P}^3 = \begin{pmatrix} 0.29 & 0.29 & 0.42 & 0 \\ 0.23 & 0.23 & 0.36 & 0.18 \\ 0.25 & 0.17 & 0.41 & 0.17 \\ 0.34 & 0.21 & 0.28 & 0.17 \end{pmatrix}, \quad \overline{P}^4 = \begin{pmatrix} 0.21 & 0.17 & 0.29 & 0.33 \\ 0.33 & 0.28 & 0.22 & 0.17 \\ 0.2 & 0.4 & 0.33 & 0.07 \\ 0.09 & 0.3 & 0.39 & 0.22 \end{pmatrix}.$$

The collective fuzzy preference relation obtained by using the P-IOWA operator guided by the same fuzzy linguistic quantifier "*most of*", and the expression (4) applied to the relative preference contribution of the values to be aggregated, is

$$P^c = \begin{pmatrix} 0.5 & 0.45 & 0.69 & 0.68 \\ 0.61 & 0.5 & 0.71 & 0.5 \\ 0.36 & 0.49 & 0.5 & 0.18 \\ 0.88 & 0.64 & 0.88 & 0.5 \end{pmatrix}.$$

Note 7. The collective preference relation obtained by the application of the P-IOWA operator does not verify the reciprocity property. This is due to the fact that this P-IOWA operator behaves as an OWA operator which normally does not maintain the reciprocity property [2].

Note 8. The P-IOWA operator when applied to a set of fuzzy preference relations may be seen as a family of IOWA operators, each one of them using a set of the elements from the same cell of a particular fuzzy preference relation as its order inducing values. However, if the P-IOWA operator is applied to just one fuzzy preference relation, as it may be the case of deriving a choice degree for each alternative as done in [1], gives the same result that the OWA operator. Indeed, in this case the aggregation values are calculated row by row and therefore, as was pointed out in Proposition 1, the relative preference values are related to the preference values by the increasing function $f(p_{ir}) = \frac{p_{ir}}{C_i}$ where C_i is the total sum of the elements of row i, $C_i = \sum_j p_{ij}$.

4. A selection process for GDM problems based on fuzzy majority and IOWA operators

Following the choice scheme proposed in [1], i.e., *aggregation* followed by *exploitation*, we design a selection process for GDM problems based on fuzzy majority and the IOWA operators presented in this paper.

As we aforementioned, when aggregating a set of 2-tuples using IOWA operators, ties may appear among the values of the ordering variable and the aggregated values could be different according to the procedure applied. This was not a problem when using OWA operators where ties do not affect the aggregated values. In the case of aggregating fuzzy preference relations, when using IOWA operators, we propose a sequential procedure for GDM problems, different to the one proposed by Yager and Filev in [27].

4.1. A procedure to deal with ties using IOWA operators

The procedure is applied in three steps, as follows:

1. If the GDM problem is heterogeneous then the I-IOWA operator is applied; if not the C-IOWA operator is applied.
2. If an I-IOWA operator has been applied in 1 then the ordering of the equally important information is induced based upon their respective consistency index values (C-IOWA). If a C-IOWA operator has been

applied in 1 then the ordering of the equally consistent information is induced based upon their respective relative preference values (P-IOWA).
3. Finally, if ties are still present then the ordering is induced based upon the values of the argument variable, i.e., the usual OWA operator is applied.

4.2. Selection process

We suppose that we have a group of experts, $E = \{e_1, \ldots, e_m\}$, which provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, by means of the fuzzy preference relations, $\{P^1, \ldots, P^m\}$, $P^k = [p^k_{ij}]$, $p^k_{ij} \in [0,1]$, which are additive reciprocal, i.e., $p^k_{ij} + p^k_{ji} = 1,\ \forall i,j,k$.

1. Aggregation phase: Obtaining a collective fuzzy preference relation.
From the set of m individual fuzzy preference relation $\{P^1, \ldots, P^m\}$ we derive the collective preference relation $P^c = (p^c_{ij})$. Each value p^c_{ij} is calculated as follows:
(a) If the GDM problem is heterogeneous then
$$p^c_{ij} = \Phi^I_Q(\langle \mu_I(e_1), p^1_{ij}\rangle, \ldots, \langle \mu_I(e_m), p^m_{ij}\rangle).$$
(b) If the GDM problem is homogeneous then
$$p^c_{ij} = \Phi^C_Q(\langle 1 - CI^1, p^1_{ij}\rangle, \ldots, \langle 1 - CI^m, p^m_{ij}\rangle).$$
The IOWA operator reflects the fuzzy majority calculating its weights by means of the fuzzy linguistic quantifier Q.
If ties appear then the above procedure is applied.
These collective preference values represent the preference of one alternative over another for the majority (Q) of the more important or (and) consistent experts.
2. Exploitation Phase: Choosing the best alternative(s).
At this point, in order to select the alternative(s) "best" acceptable for the majority (Q) of the experts we apply to the collective preference relation the quantifier guided dominance choice degree and obtain for every alternative, x_i, a value, $QGDD_i$, that quantify the dominance that one alternative has over all the others in a fuzzy majority sense (Q'):
$$QGDD_i = \Phi^P_{Q'}(\langle \bar{p}^c_{i1}, p^c_{i1}\rangle, \ldots, \langle \bar{p}^c_{in}, p^c_{in}\rangle),$$
using the notation as in Definition 5.
3. Finally, the solution set of alternatives of the GDM problem is
$$X^{\text{sol}} = \{x_i | x_i \in X,\ QGDD_i = \max_j QGDD_j\}.$$

Example 6. Suppose the same set of experts and alternatives of Example 4. Suppose that the importance degrees of these four experts are $\mathbf{I} = (2.4, 2.4, 1.1, 2.1)$, showing a tie between the experts e_1 and e_2. Using their respective consistency index values, $1 - CI^1 = 0.4$ and $1 - CI^2 = 0.86$ respectively, we get the final induced ordering of the four experts $\{e_2, e_1, e_4, e_3\}$. We call this aggregation operator as the importance consistency induced weighted averaging (IC-IOWA) operator.

1. The collective preference relation obtained using the fuzzy linguistic quantifier "*most of*" is
$$P^c = 0.55 \cdot P^2 + 0.23 \cdot P^1 + 0.15 \cdot P^4 + 0.07 \cdot P^3 = \begin{pmatrix} 0.5 & 0.38 & 0.65 & 0.26 \\ 0.62 & 0.5 & 0.64 & 0.43 \\ 0.35 & 0.36 & 0.5 & 0.13 \\ 0.74 & 0.56 & 0.87 & 0.5 \end{pmatrix},$$

whose elements represent the preference of one alternative over another for "*most of*" the more important and consistent experts.

2. The dominance guided choice degrees obtained using the fuzzy linguistic quantifier "*most of*", with the corresponding weighting vector $(0.5, 0.21, 0.16, 0.13)$, are:

   $$\mathbf{QGDD} = (0.52, 0.58, 0.4, 0.77).$$

   These values represent the dominance of one alternative over "*most of*" the rest, for "*most of*" the more important and consistent experts.

3. Clearly, the solution set is: $X^{\text{sol}} = \{x_4\}$.

Note 9. Although the P-IOWA operator when applied to just one fuzzy preference relation gives the same result that the regular OWA operator, this is not the case when applied to a set of fuzzy preference relations. The following exemplifies the differences between both operators:

Suppose two equally important experts provide the following preference relations:

$$P^1 = \begin{pmatrix} 0.5 & 0.6 & 0.35 & 0.8 \\ 0.4 & 0.5 & 0.25 & 0.7 \\ 0.65 & 0.75 & 0.5 & 0.95 \\ 0.2 & 0.3 & 0.05 & 0.5 \end{pmatrix}, \quad P^2 = \begin{pmatrix} 0.5 & 0.4 & 0.2 & 0.15 \\ 0.6 & 0.5 & 0.3 & 0.25 \\ 0.8 & 0.7 & 0.5 & 0.45 \\ 0.85 & 0.75 & 0.55 & 0.5 \end{pmatrix},$$

which have the same consistency index. If we aggregate them using the regular OWA operator we have the collective preference relation

$$P^c_{\text{OWA}} = \begin{pmatrix} 0.5 & 0.555 & 0.32 & 0.746 \\ 0.555 & 0.5 & 0.287 & 0.636 \\ 0.761 & 0.736 & 0.5 & 0.862 \\ 0.785 & 0.68 & 0.529 & 0.5 \end{pmatrix},$$

while if we aggregate them using the P-IOWA operator then we get:

$$\overline{P^1} = \begin{pmatrix} 0.222 & 0.267 & 0.156 & 0.356 \\ 0.216 & 0.27 & 0.135 & 0.378 \\ 0.228 & 0.263 & 0.175 & 0.333 \\ 0.19 & 0.286 & 0.048 & 0.476 \end{pmatrix}, \quad \overline{P^2} = \begin{pmatrix} 0.4 & 0.32 & 0.16 & 0.12 \\ 0.364 & 0.303 & 0.182 & 0.152 \\ 0.327 & 0.286 & 0.204 & 0.184 \\ 0.321 & 0.283 & 0.208 & 0.189 \end{pmatrix},$$

$$P^c_{\text{P-IOWA}} = \begin{pmatrix} 0.5 & 0.452 & 0.243 & 0.712 \\ 0.558 & 0.5 & 0.288 & 0.63 \\ 0.765 & 0.714 & 0.5 & 0.851 \\ 0.715 & 0.431 & 0.501 & 0.5 \end{pmatrix}.$$

We observe that the results are different. For example, the collective value p^c_{12} with the regular OWA aggregation

$$p^c_{12} = w_1 \cdot p^1_{12} + w_2 \cdot p^2_{12} = 0.77 \cdot 0.6 + 0.23 \cdot 0.4 = 0.56$$

is closer to the value provided by the first expert which is in first place in the ordering used in the aggregation; while in the P-IOWA aggregation

$$p^c_{12} = w_1 \cdot p^2_{12} + w_2 \cdot p^1_{12} = 0.74 \cdot 0.4 + 0.26 \cdot 0.6 = 0.45$$

is closer to the value provided by the second expert, which is now in first place in the ordering induce by the relative preference values associated to each one of the preference values to aggregate. Although 0.6 is greater than 0.4, 0.4 is the greatest value in its corresponding row and have a higher relative preference contribution to the total preference of the alternative x_1 for the second expert than 0.6 has for the first expert. Therefore, the use of the P-IOWA operator can be very useful in tie situations when we have equally important and equally consistent information.

5. Reciprocity and consistency properties of the collective fuzzy preference relation

In GDM models we normally assume that the fuzzy preference relations are reciprocal. However, it is well known that reciprocity is not generally preserved after aggregation is carried out in the resolution process [2]. In Example 3 the collective fuzzy preference relation obtained, using the I-IOWA operator, was reciprocal and the one obtained in Example 4, using the C-IOWA operator, was also reciprocal. However, the one obtained in Example 5 by applying the P-IOWA was not reciprocal.

In what follows, we will show that IOWA operators acting as WA operators maintain both the reciprocity and the consistency properties. On the other hand, the IOWA operators acting as OWA operators do not generally maintain these properties as shown by Example 5 and in [2], where we proved that the subclass of OWA operators do not generally maintain, in the aggregation process, the reciprocity and consistency properties.

5.1. Reciprocity property

If a group of experts, $E=\{e_1,\ldots,e_m\}$, provides preferences about the alternatives, $X=\{x_1,\ldots,x_n\}$, by means of reciprocal fuzzy preference relations, $\{P^1,\ldots,P^m\}$, $p^k_{ij}+p^k_{ji}=1,\ \forall i,j,k$, and if $\{u_1,\ldots,u_m\}$ is a set of order inducing (importance, consistency) values associated to the set of experts, then the collective preference relation, $P^c=(p^c_{ij})$ obtained by using an IOWA operator Φ_Q guided by a fuzzy linguistic quantifier Q is also reciprocal.

Indeed, on the one hand,

$$p^c_{ij}=\Phi_Q(\langle u_1,p^1_{ij}\rangle,\ldots,\langle u_n,p^m_{ij}\rangle)=\sum_{k=1}^m w_k\cdot p^{\sigma(k)}_{ij},$$

being $\sigma:\{1,\ldots,n\}\to\{1,\ldots,n\}$ a permutation such that $u_{\sigma(k)}\geqslant u_{\sigma(k+1)},\ \forall\ k=1,\ldots,n-1$. On the other hand, it is clear that

$$p^c_{ji}=\Phi_Q(\langle u_1,p^1_{ji}\rangle,\ldots,\langle u_n,p^m_{ji}\rangle)=\sum_{k=1}^m w_k\cdot p^{\sigma(k)}_{ji}=\sum_{k=1}^m w_k\cdot(1-p^{\sigma(k)}_{ij})=1-\sum_{k=1}^m w_k\cdot p^{\sigma(k)}_{ij}=1-p^c_{ij}$$

and thus P^c verifies the reciprocity property.

5.2. Consistency property

If the set of fuzzy preference relations are additive consistent [10], i.e.,

$$p^k_{ij}+p^k_{jl}+p^k_{li}=\frac{3}{2},\quad \forall i,j,l\in\{1,\ldots,n\},\quad k\in\{1,\ldots,m\}$$

and $P^c=\Phi_Q(\langle u_1,P^1\rangle,\ldots,\langle u_n,P^m\rangle)$, then

$$\begin{aligned}p^c_{ij}+p^c_{jl}+p^c_{li}&=\sum_{k=1}^m w_k\cdot p^{\sigma(k)}_{ij}+\sum_{k=1}^m w_k\cdot p^{\sigma(k)}_{jl}+\sum_{k=1}^m w_k\cdot p^{\sigma(k)}_{li}=\sum_{k=1}^m w_k\cdot\left(p^{\sigma(k)}_{ij}+p^{\sigma(k)}_{jl}+p^{\sigma(k)}_{li}\right)\\&=\sum_{k=1}^m w_k\cdot\frac{3}{2}=\frac{3}{2},\end{aligned}$$

which proves the additive consistency of P^c.

Note 10. The above proof of reciprocity and consistency of the collective fuzzy preference relation is based upon the assumption that the order inducing values are unchanged.

6. Concluding remarks

In this paper we have studied the use of the IOWA operators in the aggregation of fuzzy preference relations in GDM problems. We have defined three IOWA operators: the I-IOWA operator, which applies the ordering of the argument values based upon the importance of the information sources; the C-IOWA operator, which applies the ordering of the argument values based upon the consistency of the information sources; and the P-IOWA operator, which applies the ordering of the arguments based upon the relative preference associated to each one of them. We have also given a sequential procedure to deal with ties in respect to the ordering induced by the application of one of these IOWA operators, that consists of a sequential application of the above IOWA operators. The application of this sequential procedure induces an ordering of the arguments to aggregate without ties, or in the extreme case of their presence these do not affect the aggregated result. A Selection Process for GDM Problems Based on Fuzzy Majority and IOWA Operators was presented. Finally, we have shown that the collective fuzzy preference relation verifies the reciprocity and consistency properties under the assumption that the order inducing values are unchanged.

The main advantage of this proposal is the inclusion of particular orderings of fuzzy preference relations in the aggregation phase of heterogeneous and/or homogeneous GDM problems. In this way, we can take into consideration the reliability of the sources of information (experts) according to either their importance degrees (heterogeneous context) or consistency levels (homogeneous context) using IOWA operators.

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